A PROOF OF THE THEOREM CONCERNING ARTIFICIAL SINGULARITIES

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In the January number of the *Bulletin of the American Mathematical Society*,* Professor Landau has supplied data lacking in certain fallacious proofs of the well-known theorem which states that if $f(z)$ is single-valued and analytic at all points of the neighborhood of the point c, exclusive of c, and remains finite throughout this neighborhood, it has at most an artificial singularity at c; i. e., with a suitable definition of $f(c)$ the function $f(z)$ will be analytic at c.†

I wish to point out in this note a simple proof of this theorem which does not assume, as does Professor Landau's paper, the existence of $\lim_{z=c} f(z)$.

The proof makes use of an auxiliary function $\psi(z)$ defined by the equations

$$\psi(z) = (z-c)^2 f(z) \qquad (z \neq c), \qquad \psi(c) = 0.‡$$

We have

$$\psi'(z) = 2(z-c)f(z) + (z-c)^2 f'(z) \qquad (z \neq c),$$
$$\psi'(c) = 0.$$

Thus $\psi(z)$ has a derivative at c as well as throughout its neighborhood. Though Goursat's theorem § makes it unnecessary to prove $\psi'(z)$ continuous

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* Vol. 12, pp. 155–156.

† For references and a discussion of various proofs of this theorem see Professor Osgood's paper, "Some points in the elements of the theory of functions," *Bull. Amer. Math. Soc.*, vol. 2 (1896), pp. 296–302.

‡ The auxiliary function considered by Professor Landau and used in the above-mentioned incorrect proofs was the function $\phi(z)$ defined as follows:

$$\phi(z) = (z-c)f(z) \qquad (z \neq c), \qquad \phi(c) = 0.$$

§ *Trans. Amer. Math. Society*, vol. 1 (1900), pp. 14–16.

at c, we may note that this property is easily deduced from the formula, valid when $|z-c|$ is sufficiently small,

$$f'(z) = \frac{1}{2\pi i}\int_C \frac{f(t)\,dt}{(t-z)^2},$$

where C is a circumference about z of radius $\frac{1}{2}|z-c|$. If $M > |f(z)|$ throughout the neighborhood of c, exclusive of c, we have

$$|f'(z)| < \frac{2M}{|z-c|};$$

hence

$$\lim_{z=c} \psi'(z) = \psi'(c) = 0.$$

But it can be shown, either by the use of Taylor's series or directly from Cauchy's formula written in the form

$$\begin{aligned}\psi(z) &= \frac{1}{2\pi i}\int \frac{\psi(t)\,dt}{t-z} \\ &= \frac{1}{2\pi i}\int \frac{\psi(t)\,dt}{t-c} + \frac{z-c}{2\pi i}\int \frac{\psi(t)\,dt}{(t-c)^2} + \frac{(z-c)^2}{2\pi i}\int \frac{\psi(t)\,dt}{(t-z)(t-c)^2} \\ &= \psi(c) + (z-c)\,\psi'(c) + (z-c)^2\,\Psi(z),^*\end{aligned}$$

that on account of the relation $\psi(c) = \psi'(c) = 0$, $\psi(z)$ can be expressed as the product of $(z-c)^2$ and a function $\Psi(z)$ analytic at the point c as well as throughout its neighborhood. We have therefore only to add the definition $f(c) = \Psi(c)$ to make $f(z)$ identical with $\Psi(z)$ throughout the neighborhood of c, inclusive of c.

EVANSTON, ILLINOIS,
JANUARY, 1906.

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* The corresponding general form of Taylor's series with the remainder, although capable of wide application, has received little notice in treatises on the theory of functions.